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On Determining the Weight for
Obtaining A Large Number of Items

by

Kai Fun Yu *

University of South Carolina
Statistics Technical Report No. 126
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DEPARTMENT OF STATISTICS

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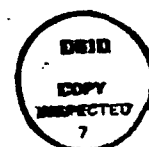## Columbia, South Carolina 29208

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Abstract

A simple procedure is proposed to determine a sample size for estimating the mean weight of items in a problem of obtaining a batch of a large number of items. Suppose it is desired to obtain a large number N_s of items for which individual counting is impractical, but one can demand a batch to weigh at least w units and hope that the number of items in the batch is close to the desired number N_s. If the items have mean weight θ, it is reasonable to have w equal to θN_s when θ is known. When θ is unknown, one can take a sample of size n, not bigger than N_s, estimate θ by a good estimator θ_n and set w equal to $\theta_n N_s$. The proposed procedure determines the sample size to be the integer closest to $\rho C N_s$, where C is a function of the cost coefficients if the coefficient of variation ρ is known. It is shown to be optimal in some sense. If ρ is unknown, a simple sequential procedure is proposed for which the average sample number is shown to be asymptotically equal to the optimal fixed sample size. When the weights are assumed to have a gamma distribution given ϕ and ϕ has a prior inverted gamma distribution, the optimal sample size in some sense can be found to be the nonnegative integer closest to $\rho C N_s + \rho^2 A(\rho C-1)$, where A is a known constant given in the prior distribution.

Key Words:

1. Introduction.

Suppose it is desired to obtain a batch of a large number N_s of items and it is impractical to count them individually. However, it is possible to require the batch to have at least a certain weight w and hope that the number N of items in the batch is close to N_s. The problem then is to determine w. If the weight of the items is constant and is equal to θ each, then $w = \theta N_s$. When a batch with weight w is delivered, it will contain exactly N_s items. However, if the weights of the items follow a distribution with mean θ and nonzero variance σ^2, then even if θ is known, the reasonable weight $w = \theta N_s$ will not yield a batch of exactly N_s items. Instead, the number N of items is determined by

$$N = \inf \{k \geq 1 : X_1 + \ldots + X_k \geq w\}, \tag{1.1}$$

where the X's are the weights of the items and the actual total weight $w^* = X_1 + \ldots + X_N$. Even in this case of known mean θ, N so determined will incur a mean squared error of

$$E(N-N_s)^2 \tag{1.2}$$

which is not zero unless $\sigma^2 = 0$. If the mean weight θ of each item is unknown, it is possible to take a sample $\{X_1, \ldots, X_n\}$ of size n (not bigger than N_s) to have a good estimate $\hat{\theta}_n$ of θ and determine

$$w = X_1 + \ldots + X_n + (N_s - n)\hat{\theta}_n. \tag{1.3}$$

(The case of known θ corresponds to that n=0, and $\hat{\theta}_n = \theta$.) The original problem has then become that of determining (a) a sample size n, (b) a good estimate $\hat{\theta}_n$ of θ and (c) the total weight w demanded. Guttman and Menzefricke

(1986) have invest.gated this problem by assuming that the distribution of the X's given θ is normal with unknown mean θ and known variance σ^2 and θ has a known prior normal distribution. In this case, they have attempted to choose n so that

$$E(K_e(N-N_s)^2 + K_s n) \tag{1.4}$$

is minimized, where K_e and K_s are the known cost coefficients, $\hat{\theta}_n$ is the posterior mean and w is $X_1 + \ldots + X_n + (N_s - n)\hat{\theta}_n$. They have applied a fundamental equation in renewal theory given as (2.1) in their paper to compute (1.4). However, since the X's are assumed to have a normal distribution, (2.1) would not hold. Although it is well-known (cf. Feller II, p.372) that the asymptotic distribution of N given w is normal with mean w/θ and variance $w\sigma^2/\theta^3$ as N_s becomes large, it is not clear how the development in Guttman and Menzefricke (1986) is theoretically justified. Moreover, even if the development has been fully justified, the determination of the optimal sample size n has not been so easy; it would have to involve some quite complicated computation.

In this note, we shall develop a treatment of the aforementioned problem, using the asymptotic renewal theory. As a result, the determination of a sample size will be very easy and simple arithmetic under various scenarios. Specifically, we shall make the following standing assumptions. Let X_1, X_2,... be independent and identically distributed random variables (representing the individual weights of the items and not necessarily normal) with positive mean θ and variance σ^2. For each nonnegative integer n, let w be defined as in (1.3) and N be defined as in (1.1). The problem is to choose n and $\hat{\theta}_n$ in (1.3)

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so that (1.4) is minimized in some sense. We shall specify in what sense n is chosen optimally in each of the following cases being studied: (a) the mean θ is a known constant, (b) the mean θ is an unknown constant, (c) the mean θ is an unknown random variable having a known distribution and (d) the mean θ is an unknown random variable having an unknown distribution.

2. The mean θ is a known constant.

In this section, instead of assuming normality for the distribution of the X's, we shall assume that the X's follow a nonparametric distribution and we know the mean θ. In this case, the obvious choice of n is zero and the natural choice of $\hat{\theta}_n$ is θ. Then $w = \theta N_s$, and N becomes the smallest integer k such that $X_1+...+X_k$ is at least θN_s. The exact computation of $E(K_e(N-N_s)^2 + K_s n)$ is impossible. However, by the well-known renewal theory (cf. Chow et al (1979)) as N_s becomes large,

$$E(K_e(N-N_s)^2) \simeq K_e N_s (\sigma/\theta)^2. \tag{2.1}$$

We shall consider the right-hand-side of (2.1) as an inherent fixed cost which cannot be eliminated even though we may know θ.

3. The mean θ is an unknown constant.

Again, we assume the X's follow a nonparametric distribution; but this time even the mean θ is unknown. Suppose a sample $\{X_1,...,X_n\}$ of fixed sample size n is taken. A reasonable choice of $\hat{\theta}_n$ is $(X_1+...+X_n)/n$, which is also the nonparametric maximum likelihood estimate. Then

$$w = X_1+...+X_n + (N_s - n)\hat{\theta}_n = \hat{\theta}_n N_s, \tag{3.1}$$

and

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$$N_n = N-n = \inf\{k \geq 1: X_{n+1} + \ldots + X_{n+k} \geq \hat{\theta}_n (N_s-n)\}.$$

The risk function is

$$E(K_e(N-N_s)^2 + K_s n) \tag{3.2}$$

$$= K_e E(N_n - (N_s-n)\frac{\hat{\theta}_n}{\theta})^2 + K_e(N_s - n)^2 E(\frac{\hat{\theta}_n-\theta}{\theta})^2 + K_s n$$

$$+ 2K_e(N_s-n)E(N_n - (N_s-n)\frac{\hat{\theta}_n}{\theta})(\frac{\hat{\theta}_n-\theta}{\theta})$$

$$= K_e E(N_n - (N_s-n)\frac{\hat{\theta}_n}{\theta})^2 + K_e (N_s-n)^2 \frac{\sigma^2}{n\theta^2} + K_s n$$

$$+ 2K_e(N_s-n)E(N_n-(N_s-n)\frac{\hat{\theta}_n}{\theta})(\frac{\hat{\theta}_n-\theta}{\theta}).$$

By the renewal theory, as N_s becomes large

$$E(N_n - (N_s-n)\frac{\hat{\theta}_n}{\theta})^2 \simeq (N_s-n)(\sigma/\theta)^2, \tag{3.3}$$

and

$$(N_s-n)E(N_n - (N_s-n)\frac{\hat{\theta}_n}{\theta})(\frac{\hat{\theta}_n-\theta}{\theta}) = o\,(\frac{N_s-n}{\sqrt{n}}). \tag{3.4}$$

Therefore the risk function is asymptotically

$$K_e N_s(\sigma/\theta)^2 + K_e \frac{(N_s-n)^2}{n} (\sigma/\theta)^2 + (K_s - K_e (\sigma/\theta)^2)n \tag{3.5}$$

$$= -K_e N_s(\sigma/\theta)^2 + K_e(\sigma/\theta)^2 \frac{N_s^2}{n} + K_s n.$$

Since the inherent fixed cost in (3.5) does not involve the sample size n, we shall consider the following transformed risk function (with n not greater than N_s)

$$R_n = K_e(\sigma/\theta)^2 \frac{N_s^2}{n} + K_s n \tag{3.6}$$

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which is minimized by taking n_* to be the integer closest to

$$\left(\frac{K_e}{K_s}\right)^{1/2} \frac{\sigma}{\theta} N_s, \tag{3.7}$$

and the transformed risk function is approximately

$$2 \frac{\sigma}{\theta} N_s ((K_e K_s)^{1/2}). \tag{3.8}$$

Now if the coefficient of variation σ/θ is known, then (3.7) is computable, and one simply takes a sample of size n_* which is optimal in the sense of minimizing the transformed risk function (3.6), and determines an additional weight of $(N_s - n_*)(X_1 + \ldots + X_{n_*})/n_*$. If the coefficient of variation σ/θ is also unknown, then one can determine the sample sequentially by

$$\tau = \inf\left\{n \geq m: n \geq \left(\frac{K_e}{K_s}\right)^{1/2} \frac{\hat{\sigma}_n}{\hat{\theta}_n} N_s\right\} \tag{3.9}$$

where m is a positive integer greater than one and

$$\hat{\theta}_n = (X_1 + \ldots + X_n)/n \text{ and} \tag{3.10}$$

$$\hat{\sigma}_n^2 = \frac{1}{n} \sum_1^n (X_i - \hat{\theta}_n)^2. \tag{3.11}$$

Then

$$\tau = \inf \left\{n \geq m: \frac{\hat{\theta}_n}{\sigma} + \frac{\hat{\theta}_n(\sigma - \hat{\sigma}_n)}{\sigma \hat{\sigma}_n} \geq \frac{1}{\lambda n}\right\} \tag{3.12}$$

where $\lambda = \frac{1}{N_s} (K_s/K_e)^{1/2}$. That is, take a sample of size τ and determine the additional weight of $(N_s - \tau)(X_1 + \ldots + X_\tau)/\tau$. By the strong law of large numbers, it is straightforward to show that as N_s becomes large, provided the distribution of X is continuous or m goes to infinity as N_s but at a slower pace,

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$$\tau \doteq n_*, \text{ a.s.} \tag{3.13}$$

that is; the ratio of the sample size τ over the optimal sample size goes to one almost surely. The following theorem on the average sample number is also true.

Theorem. $E\tau \doteq n_*$ as N_s goes to infinity.

That is the average sample size of the sequential procedure is equal to the optimal sample size to the first order. The precise conditions on the X's and the proof will be given in the Appendix, as the proof is rather technical.

4. The mean θ is an unknown random variable having a known distribution

In this section, we shall assume that given θ, the X's have a parametric density and the coefficient of variation $\rho = \sigma/\theta$ is known. Specifically, given θ, let the probability density of the X's be

$$f(x;\theta) = \begin{cases} \dfrac{\alpha(\alpha x)^{\alpha-1} e^{-\alpha x/\theta}}{\theta^{\alpha}\Gamma(\alpha)}, & x > 0 \\ \\ 0, & x \le 0 \end{cases} \tag{4.1}$$

where $\alpha = 1/\rho^2$, and θ has a density

$$\pi(\theta) = \begin{cases} \dfrac{b^a e^{-b/\theta}}{\theta^{a+1}\Gamma(a)}, & \theta > 0 \\ \\ 0, & \theta \le 0 \end{cases} \tag{4.2}$$

where a and b are known positive constants. Following the development as in Section 3, we have an asymptotic risk equal to

$$K_e \frac{N_s-n}{\alpha} + K_e (N_s-n)^2 E(\frac{\hat{\theta}_n - \theta}{\theta})^2 + K_s n. \tag{4.3}$$

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An obvious choice of $\hat{\theta}_n$ is

$$\hat{\theta}_n = \frac{E(\frac{1}{\theta}|X_1,\ldots,X_n)}{E(\frac{1}{\theta^2}|X_1,\ldots,X_n)} \tag{4.4}$$

$$= \frac{\alpha(X_1+\ldots+X_n) + b}{n\alpha + a+1},$$

and the expression in (4.3) becomes

$$K_e \frac{N_s-n}{\alpha} + K_e(N_s -n)^2 \frac{1}{n\alpha+a+1} + K_s n, \tag{4.5}$$

which is minimized by taking n_* to be the nonnegative integer closest to

$$\left(\frac{K_e}{K_s}\right)^{\frac{1}{2}} \rho\, N_s + (a+1) \left(\left(\frac{K_e}{K_s}\right)^{\frac{1}{2}} \rho^3 - \rho^2 \right). \tag{4.6}$$

And the total weight can be determined by

$$w = X_1 +\ldots+X_{n_*} + (N_s-n_*)\, \hat{\theta}_{n_*}. \tag{4.7}$$

5. <u>The mean θ is an unknown random variable having an unknown distribution.</u>

We shall make the same assumption as in Section 4 except that a and b in the prior distribution are unknown and that ρ may not be known. In this case, the optimal sample size (4.6) can not be computed and the estimate (4.4) can not be evaluated. If it is known that a is small (small a corresponds to large variance of the prior distribution), then (4.6) is quite close to

$$\left(\frac{K_e}{K_s}\right)^{\frac{1}{2}} \rho N_s \tag{5.1}$$

We recommend a sample size of an integer n closest to (5.1) and estimate θ by $\hat{\theta}_n$ = $(X_1 +\ldots+ X_n)/n$ and determine the weight by

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$$N_s(X_1+\ldots+X_n)/n. \tag{5.2}$$

If ρ is also unknown, the sequential procedure as described in Section 3 is recommended. That is, the sample size τ is

$$\tau = \inf\ \{n \geq m:\ \hat{\theta}_n/\hat{\sigma}_n \geq \left(\frac{K_e}{K_s}\right)^{\frac{1}{2}} N_s/n\}, \tag{5.3}$$

where m is a positive integer greater than one,

$$\hat{\theta} = (X_1+\ldots+X_n)/n, \text{ and} \tag{5.4}$$

$$\hat{\sigma}_n^2 = 1/n \sum_1^n (X_i - \hat{\theta}_n)^2. \tag{5.5}$$

Then the weight determined is

$$w = N_s(X_1+\ldots+X_\tau)/\tau. \tag{5.6}$$

If a is not known to be small, then (5.1) will not be close to (4.6). There does not seem to be much one can do except in the case when there are previous data available. In this case, an empirical Bayes approach can be applied. A study of such an approach will be conducted elsewhere.

6. An example.

We shall present a table of optimal sizes for different values of ρ, a, b, K_s, K_e and N_s. The optimal sample size in the Bayes columns is computed from (4.6), that is, the nonnegative integer closest to

$$\left(\frac{K_e}{K_s}\right)^{\frac{1}{2}} \rho N_s + (a+1)\rho^2 \left[\rho\left(\frac{K_e}{K_s}\right)^{\frac{1}{2}} - 1\right] \tag{6.1}$$

and the optimal sample size for the nonparametric column is computed from (3.7), the integer closest to

$$\left(\frac{K_e}{K_s}\right)^{\frac{1}{2}} \rho N_s \qquad\qquad (6.2)$$

We have chosen the values such that the first part corresponds quite closely to that reported by Guttman and Menzefricke (1986). Notice that the optimal sample sizes are quite unstable in the Bayes case as a varies. Of course, when a is large, it corresponds to small variance in the prior distribution which amounts to saying that the mean is known quite precisely and thus less sample size needs to be taken. The nonparametric column also gives the asymptotic average sample numbers in the first order approximation when they are big.

Table 1. Optimal Sample Sizes ($N_s = 20,000$; $K_s = 1$)

	Bayes					Nonparametric
	a					
	360000	36000	3600	360	36	
A. $\rho = 1/60$						
$K_e=100$	3250	3325	3332	3333	3333	3333
1	235	323	332	333	333	333
.01	0	23	32	33	33	33
.0001	0	0	2	3	3	3
B. $\rho = 1/30$						
$K_e=100$	6400	6640	6664	6666	6667	6667
1	280	628	663	666	667	667
.01	0	27	63	66	67	67
.0001	0	0	3	6	7	7
C. $\rho = 1/15$						
$K_e=100$	12800	13280	13328	13333	13333	13333
1	0	1184	1318	1332	1333	1333
.01	0	0	117	132	133	133
.0001	0	0	0	12	13	13

Appendix.

Theorem: Let $X, X_1, X_2, \ldots$ be independent and identically distributed random variables with positive mean θ and finite nonzero variance σ^2. For each $n \geq 1$, let

$$\hat{\theta}_n = \frac{1}{n}(X_1 + \ldots + X_n), \tag{A.1}$$

$$\hat{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \hat{\theta}_n)^2, \text{ and} \tag{A.2}$$

$$\tau = \inf \{ n \geq m: \frac{\hat{\theta}_n}{\sigma} + \frac{\hat{\theta}_n(\sigma - \hat{\sigma}_n)}{\sigma\hat{\sigma}_n} \geq \frac{1}{\lambda n} \} \tag{A.3}$$

where m is a positive integer bigger than one, and λ is a positive constant. Assume the distribution of X is continuous. If $E(X^2)^p < \infty$ for some $p \geq 1$, then

(i) $\{(\lambda\tau)^p, 0 < \lambda \leq 1\}$ is uniformly integrable, and

(ii) $\quad E|\lambda\tau - \frac{\sigma}{\theta}|^p \to 0$ as $\lambda \to 0$.

Proof: Consider

$$R_n = \frac{\hat{\theta}_n(\sigma - \hat{\sigma}_n)}{\sigma\hat{\sigma}_n} \tag{A.4}$$

$$= \frac{\hat{\theta}_n}{\sigma\hat{\sigma}_n(\sigma + \hat{\sigma}_n)} \frac{1}{n}\sum_{i=1}^{n}(\sigma^2 - (X_i - \theta)^2) + \frac{\hat{\theta}_n(\hat{\theta}_n - \theta)^2}{\sigma\hat{\sigma}_n(\sigma + \hat{\sigma}_n)}.$$

For any small $\varepsilon > 0$, let

$$\alpha = \inf \{n \geq 1: |\hat{\theta}_n - \theta| \leq \varepsilon\} \tag{A.5}$$

Then since $E(X^2)^p < \infty$, by Lemma 3 in Chow et al (1983), $E\alpha^{2p} < \infty$. Make copies of α and denote them by $\alpha^{(1)}, \alpha^{(2)}, \ldots$ and let $\alpha_n = \alpha^{(1)} + \ldots + \alpha^{(n)}$. For each $n \geq 1$, let

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$$Y_n = (\sigma^2 - (X_{\alpha_{n-1}+1} - \theta)^2) + \ldots + (\sigma^2 - (X_{\alpha_n} - \theta)^2) \tag{A.6}$$

where $\alpha_0 = 0$. Since $E(X^2)^p < \infty$ and $E\,\alpha^{2p} < \infty$, by Lemma 2(ii) in Chow et al (1983), $E|Y_1|^p < \infty$. Define

$$\beta = \inf\ \{n \geq 1:\ \ |Y_1 + \ldots + Y_n| \leq \varepsilon \alpha_n\} \tag{A.7}$$

Since $E|Y_1|^p < \infty$, by Lemma 3 in Chow et al (1983), $E\beta^p < \infty$. Let $t = \alpha_\beta$. Since $E\alpha^{2p} < \infty$ and $E\beta^p < \infty$, by Lemma 2(iii) in Chow et al (1983), $Et^p < \infty$. And t is a stopping time. Make copies of t and denote by $t^{(1)}$, $t^{(2)}$, $\ldots$ and let $t_n = t^{(1)} + \ldots + t^{(n)}$. Then for each $n \geq 1$,

$$\hat{\sigma}^2_{t_n} \geq \sigma^2 - \varepsilon - \varepsilon^2. \tag{A.8}$$

Hence

$$|R_{t_n}|$$
$$\leq \frac{(\theta+\varepsilon)\ (\varepsilon+\varepsilon^2)}{\sigma(\sigma^2 - \varepsilon-\varepsilon^2)^{\frac{1}{2}}\ (\sigma + (\sigma^2 - \varepsilon-\varepsilon^2)^{\frac{1}{2}})}. \tag{A.9}$$

By Theorem 1 of Yu (1986), $\{(\lambda\tau)^p, 0 < \lambda \leq 1\}$ is uniformly integrable. By the strong law of large numbers

$$\lambda\tau - \frac{\sigma}{\theta} \to 0, \text{ almost surely,} \tag{A.10}$$

as λ goes to zero. Hence as λ goes to zero

$$E\left|\lambda\tau - \frac{\sigma}{\theta}\right|^p \to 0; \tag{A.11}$$

in particular, if $\lambda = \frac{1}{N_s}\ (K_s/K_e)^{\frac{1}{2}}$,

$$E\tau \simeq \left(\frac{K_e}{K_s}\right)^{\frac{1}{2}} \frac{\sigma}{\theta}\ N_s. \tag{A.12}$$

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References

Chow, Y. S., Hsiung, C. A. and Lai, T. L. (1979). Extended renewal theory and moment convergence in Anscombe's theorem. Annals of Probability, 7, 304–318.

Chow, Y. S., Hsiung, C. A., and Yu, K. F. (1983). A renewal theorem and its applications to some sequential procedures. Zeitschrift für Wahrschein-lichkeitstheorie und verwandte Gebiete, 64, 241–250.

Feller, W. (1971). An Introduction to Probability and Its Application. Volume II. John Wiley and Sons, Inc., New York.

Guttman, I. and Menzefricke, U. (1986). Counting by weighing: an approach using renewal theory. Journal of the American Statistical Association, 81, 129–131.

Nelson, P. R. (1983). Counting a large number of items using the weight of a small sample. Technometrics, 25, 173–177.

Yu, K. F. (1986). A note on a renewal theorem for a moving average process. Bulletin Institute Mathematics Acedemia Sinica, 14, 411–416.

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